Solution 9

1. Consider Theorem 3.4. Show that the function Φ maps $\overline{B_r(x_0)}$ to $\overline{B_R(y_0)}$ has an inverse, that is, there is a continuous function G in $B_R(y_0)$ back to $B_r(x_0)$ satisfying $\Phi(G(y)) = y$ for all $y \in \overline{B_R(y_0)}$.

   Solution. Theorem 3.4 asserts that for each $y \in \overline{B_R(y_0)}$ there is a unique $x \in \overline{B_r(x_0)}$ satisfying $\Phi(x) = y$. The correspondence $y \mapsto x$ defines a map G that satisfies $\Phi(G(y)) = y$. (Note that G may not map $B_R(y_0)$ onto $B_r(x_0)$. More precisely speaking, G is the inverse map of the restriction of Φ to the set $G(\overline{B_R(y_0)})$.) It remains to show that G is continuous. Indeed, let $y_1, y_2 \in \overline{B_R(y_0)}$, we have $\Phi(G(y_i)) = G(y_i) + \Psi(G(y_i)) = y_i, i = 1, 2$. It follows that

$$\begin{aligned} \|G(y_1) - G(y_2)\| &= \|y_1 - y_2 - \Psi(G(y_1) + \Psi(G(y_2))\| \\ &\leq \|y_1 - y_2\| + \|\Psi(G(y_1) - \Psi(G(y_2))\| \\ &\leq \|y_1 - y_2\| + \gamma\|G(y_1) - G(y_2)\| \ . \end{aligned}$$

   which implies

$$\|G(y_1) - G(y_2)\| \leq \frac{1}{1-\gamma}\|y_1 - y_2\| \ , \quad y_1, y_2 \in \overline{B_R(y_0)} \ .$$

   That is, G is Lipschitz continuous.

   Note. The same trick has appeared in the proof of the Inverse Function Theorem. Here we need to use the norm $\|\cdot\|$ to replace the Euclidean norm there, and that is it. In the Inverse Function Theorem we can further study the differentiability of the inverse map, but now we cannot do it here. Why? We have not yet considered the differentiable property in a normed space!

2. Consider the function

$$h(x, y) = (x - y^2)(x - 3y^2), \quad (x, y) \in \mathbb{R}^2.$$

   Show that the set $\{(x, y) : \ h(x, y) = 0\}$ cannot be expressed as a local graph of a C^1-function over the x or y-axis near the origin. Explain why the Implicit Function Theorem is not applicable.

   Solution. The Jacobian matrix of h is singular at $(0, 0)$, hence the Implicit Function Theorem cannot apply. Indeed, $h(x, y) = 0$ means either $x - y^2 = 0$ or $x - 3y^2 = 0$. The solution set of $\{(x, y) : \ h(x, y) = 0\}$ consisting of two different parabolas passing the origin.

3. Consider the mapping from $\mathbb{R}^2$ to itself given by $f(x, y) = x - x^2, \ g(x, y) = y + xy$. Show that it has a local inverse at $(0, 0)$. And then write down the inverse map so that its domain can be described explicitly.

   Solution. Let $u = x - x^2, v = y + xy$. The Jacobian determinant is 1 at $(0, 0)$ so there is an inverse in some open set containing $(0, 0)$. Now we can describe it explicitly as follows. From the first equation we have

$$x = \frac{1 \pm \sqrt{1 - 4u}}{2}.$$

From $u(0,0)=0$ we must have

$$x=\frac{1-\sqrt{1-4u}}{2} \ .$$

Then

$$y=\frac{v}{1+x}=\frac{2v}{1-\sqrt{1-4u}}.$$

We see that the largest domain in which the inverse exists is $\{(u,v):u\in(0,1/4),v\in\mathbb{R}\}$.

In the following the Initial Value Problem (IVP) refers to $x'=f(t,x),\ x(t_0)=x_0$, where f satisfies the Lipschitz condition in some rectangle containing (t_0,x_0) in its interior, see Notes for details.

4. Solve the (IVP) for $f(t,x)=\alpha t(1+x^2),\alpha>0,\ t_0=0$, and discuss how the interval of existence changes as α and x_0 vary.

   Solution. The solution is given by

   $$x(t)=\tan(\tan^{-1}x_0+\alpha t^2/2) \ ,$$

   where the tangent function is chosen so that $\tan:(-\pi/2,\pi/2)\to(-\infty,\infty)$. The (maximal) interval of existence is $(-a,a)$ where

   $$a=\frac{1}{\alpha}(\pi-2\tan^{-1}x_0) \ .$$

   We see that for fixed α, the interval shrinks as x_0 increases, and for fixed x_0, it shrinks too as α increases. The maximal interval of existence depends on f,t_0 and x_0 in a complicated manner.

5. Optional. Deduce Picard-Lindelöf Theorem based on the ideas of perturbation of identity. Hint: Take a particular

   $$y=\int_{t_0}^t f(t,x_0)dt$$

   in the relation $x+\Psi(x)=y$.

   Solution. Write the integral form of (IVP) as

   $$x(t)-x_0-\int_{t_0}^t (f(s,x(s))-f(s,x_0))ds=\int_{t_0}^t f(s,x_0)ds \ .$$

   Define $Tx(t)=\Psi(x)+y$, where

   $$\Psi(x)=-x_0-\int_{t_0}^t (f(s,x(s))-f(s,x_0))ds \ .$$

   Let

   $$X=\{x\in C[t_0-a',t_0+a']:|x(t)-x_0|\le b\}$$

   where $a'=\min\{a,b/M,1/L\}$ as before. We first claim, when $a'\le b/M$, T maps X to itself. Indeed,

   $$|Tx(t)-x_0|=|\int_{t_0}^t f(s,x(s))ds|\le M|t|\le b \ ,$$

by our choice. Next, claim T is a contraction on X. We have

$$|Tx_1(t) - Tx_2(t)| = |\Psi(x_1)(t) - \Psi(x_2)(t)| = \left| \int_{t_0}^{t} (f(s, x_1(s) - f(s, x_2(s)) \, ds \right| \leq L|t| \leq a'$$

by our choice. Now, apply Contraction Mapping Principle to T on X to get a unique fixed point. It is the solution of our (IVP).

6. Show that the solution to IVP belongs to C^{k+1} (as long as it exists) provided $f \in C^k$ for $k \geq 1$. In particular, $y \in C^\infty$ provided $f \in C^\infty$.

   Solution. It is an elementary fact and easy to show that the composition of two C^k-functions is again C^k. Now, from (1) we see that y is C^1 if the RHS, that is, $f(x, y(x))$ is continuous. By induction, assuming now y is C^{k+1} when f is C^k. When f is C^{k+1}, it is also C^k and so by induction hypothesis y is C^{k+1}. The RHS of (1) is the composition of twonC^{k+1}-functions and hence is also C^{k+1}. It shows that the LHS y' is C^{k+1}, that is, $y \in C^{k+2}$, done.

7. Let $f \in C(D), D = (a, b) \times (c, d)$, satisfying the Lipschitz condition. Let x_1 and x_2 be two solutions to (IVP) defined on open subintervals I and J in (a, b) respectively with $x_1(t_0) = x_2(t_0)$ at some $t_0 \in I \cap J$. Assuming that their graphs lie in D. Show that they are equal on $I \cap J$.

   Solution. Let $(\alpha, \beta) = I \cap J$. By the fundamental theorem, there is an open interval containing t_0 such that the two solutions coincide. Let $b^* = \sup\{c : \ x_1 = x_2 \text{ on } [t_0, c)\}$. By continuity $x_1(b^*) = x_2(b^*)$. If $b^* < b$, by the fundamental theorem taking b^* as the initial time, x_1 and x_2 coincide in an open interval containing b^*, contradicting the definition of b^*.

8. Fill out the details for the Picard-Lindeöf Theorem for systems (Theorem 3.12) in Notes.

   Solution. First of all, Proposition 3.11 still holds where the integral equation (3.7) becomes

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{x}(s)) \, ds \ .$$

   And we define

$$\mathcal{T}\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{x}(s)) \, ds \ ,$$

   as before. To verify $\mathcal{T}$ maps X to X (see Notes for the def of X) we have

$$\begin{aligned} |(\mathcal{T}\mathbf{x})(t) - \mathbf{x}_0| &= |\int_{t_0}^{t} \mathbf{f}(s, \mathbf{x}(s)) \, ds| \\ &\leq M|t - t_0| \\ &\leq Ma' \\ &\leq b \ , \end{aligned}$$

   so $\mathcal{T}$ maps X into X. We explain the second step in the above estimate by noting the general formula:

$$\left| \int_a^b \mathbf{f} \, ds \right| \leq \sqrt{(b-a) \int_a^b |\mathbf{f}|^2 \, ds} \ .$$

To prove,

$$
\begin{aligned}
\left|\int_a^b \mathbf{f}\,ds\right| &= \sqrt{\left(\int_a^b f_1\,ds\right)^2+\cdots+\left(\int_a^b f_n\,ds\right)^2} \\
&\le \sqrt{(b-a)\int_a^b f_1^2+\cdots+(b-a)\int_a^b f_n^2\,ds} \quad (\text{ Cauchy-Schwarz}) \\
&= \sqrt{(b-a)\int_a^b |\mathbf{f}|^2}\ .
\end{aligned}
$$

In our situation, we get

$$\left|\int_{t_0}^t \mathbf{f}(s,\mathbf{x}(s))\,ds\right| \le \sqrt{|t-t_0|\int_{t_0}^t |\mathbf{f}|^2\,ds}\ .$$

Then using $|\mathbf{f}(s,\mathbf{x}(s))| \le M \equiv \sup|\mathbf{f}|$ to get

$$\left|\int_{t_0}^t \mathbf{f}(s,\mathbf{x}(s))\,ds\right| \le M|t-t_0|\ .$$

The rest of the proof proceeds as the scalar case.

9. Show that there exists a unique solution h to the integral equation

$$h(x) = 1+\frac{1}{\pi}\int_{-1}^1 \frac{1}{1+(x-y)^2}h(y)dy,$$

in $C[-1,1]$. Also show that h is non-negative.

Solution. Let $X = C[-1,1]$ be the complete metric space we work on and set

$$(Th)(x) = 1+\frac{1}{\pi}\int_{-1}^1 \frac{1}{1+(x-y)^2}h(y)dy.$$

It is easy to check that T is continuous on X. For $h_2, h_1 \in C[-1,1]$, we have

$$
\begin{aligned}
|Th_2(x)-Th_1(x)| &= \left|\frac{1}{\pi}\int_{-1}^1 \frac{1}{1+(x-y)^2}(h_2(y)-h_1(y))dy\right| \\
&\le \frac{2}{\pi}\|h_2-h_2\|_\infty, \quad \forall x\in[-1,1].
\end{aligned}
$$

Hence T is a contraction on $C[-1,1]$, and a fixed point is ensured by Banach's Fixed Point Theorem.

Next we show that the fixed point h is non-negative. Notice that

$$\frac{1}{\pi}\int_{-1}^1 \frac{1}{1+(x-y)^2}dy = \frac{1}{\pi}[\arctan(1-x)+\arctan(1+x)] \le \frac{1}{2}, \quad x\in[-1,1].$$

From the def of h we have

$$\|h\|_\infty \le 1+\frac{1}{2}\|h\|_\infty,$$

which implies $\|h\|_\infty \leq 2$. It follows that

$$h(x) \geq 1 - \frac{1}{\pi}\int_{-1}^{1} \frac{1}{1+(x-y)^2}\|h\|_\infty dy \geq 1 - \frac{1}{2} \times 2 \geq 0,$$

h is non-negative.

Note. An alternate approach is to work on the space $Y = \{h \in C[-1,1] : \ h(x) \geq 0, \forall x\}$. From the definition of T, it is clear that T maps Y to Y. Since Y is easily shown to be a closed set in $C[-1,1]$ (hence complete), we apply the Contraction Mapping Principle directly to get a non-negative solution.